

LQ Covariance Control with Deadline Constraint and Frobenius Terminal Loss

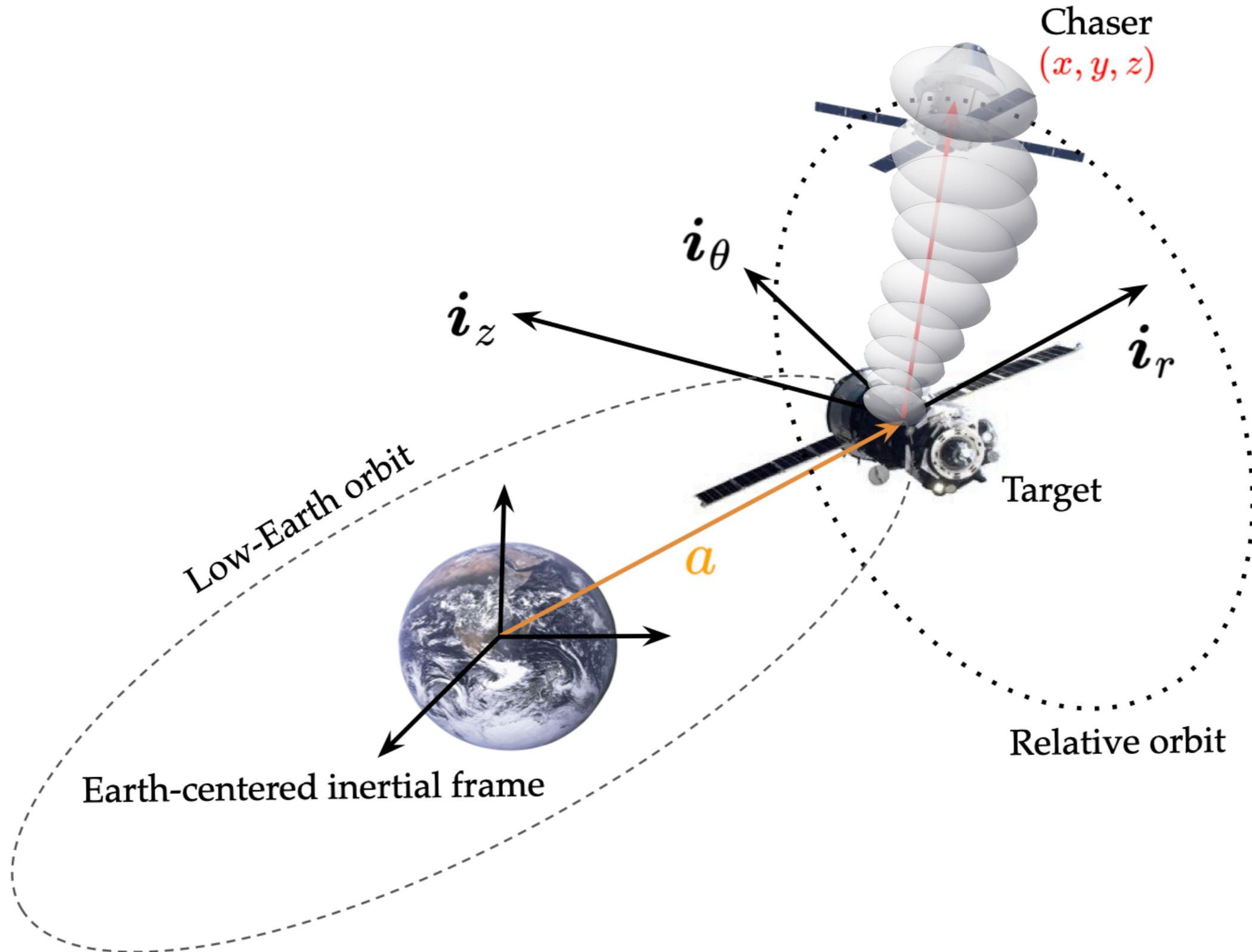
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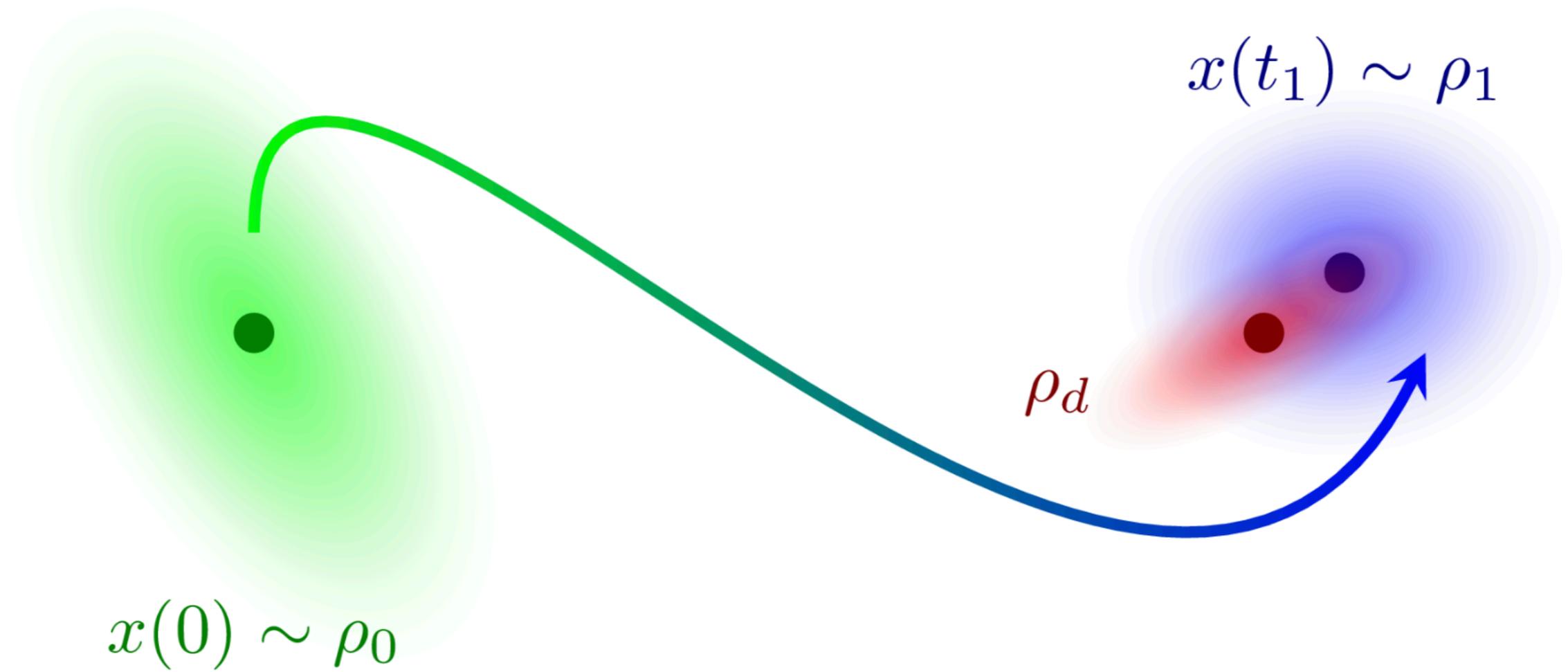
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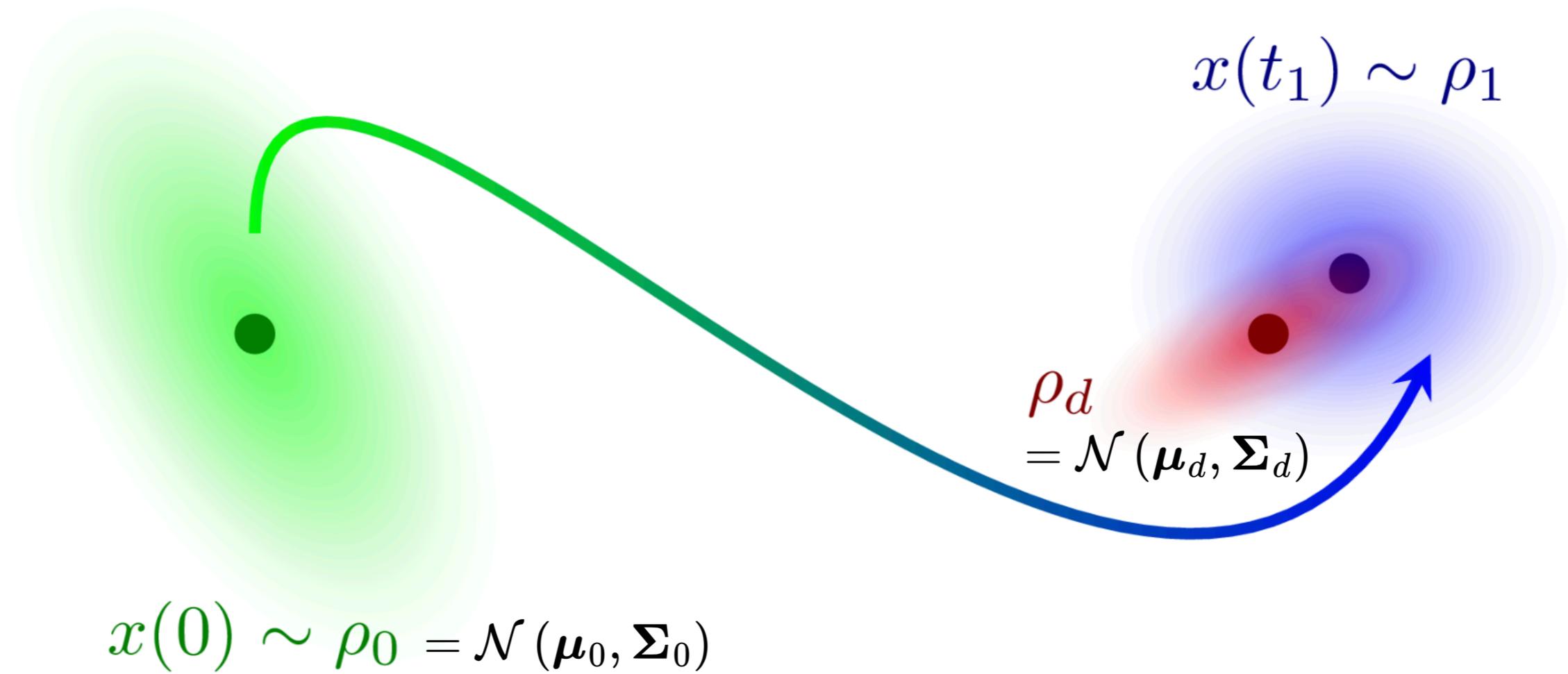
Motivation: Control of Probability Distributions



This Work: Optimal Covariance Control



This Work: Optimal Covariance Control



Current State-of-the-art

Well-investigated in **discrete time**

Balci and Bakolas, LCSS 2020

Balci, Halder and Bakolas, CDC 2021

Yin et al., ICRA 2022

Saravanos et al., IROS 2024

Morimoto and Kashima, LCSS 2024

Less investigated in **continuous time**

Halder and Wendel, ACC 2016

- derived matrix BVP for LQ case with optimal transport terminal cost + shooting method
- design of custom algorithm remains open

Hoshino, CDC 2023

- derived FBSDE for nonlinear non-Gaussian case with optimal transport terminal cost
- did not investigate numerical solution

LQ Optimal Control Problem Formulation

Linear controlled dynamics:

$$d\mathbf{x}_t = \mathbf{A}_t \underbrace{\mathbf{x}_t}_{\text{State}} dt + \mathbf{B}_t \underbrace{\mathbf{u}_t}_{\text{Input}} dt + \mathbf{B}_t \underbrace{d\mathbf{w}_t}_{\text{Process noise}}$$

Average quadratic cost-to-go:

$$\inf_{\mathbf{u}_t \in \mathcal{U}} \underbrace{\phi(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1, \boldsymbol{\mu}_d, \boldsymbol{\Sigma}_d)}_{\text{Terminal cost}} + \underbrace{\int_{t_0}^{t_1} \mathbb{E} \left(\|\mathbf{u}_t\|_2^2 + \mathbf{x}_t^\top \mathbf{Q}_t \mathbf{x}_t \right) dt}_{\text{Average cost-to-go}}$$

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Standing assumptions:

$(\mathbf{A}_t, \mathbf{B}_t) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}$ is uniformly controllable, bounded and continuous w.r.t. time t

$$\mathbf{Q}_t \succcurlyeq \mathbf{0} \quad \forall t \in [t_0, t_1], \quad \boldsymbol{\Sigma}_0, \boldsymbol{\Sigma}_d \succcurlyeq \mathbf{0}$$

Choice of the Terminal Cost $\phi(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1, \boldsymbol{\mu}_d, \boldsymbol{\Sigma}_d)$

This work: Squared Frobenius a.k.a. Hilbert-Schmidt norm

$$\frac{1}{2} \|\boldsymbol{\mu}_1 - \boldsymbol{\mu}_d\|_2^2 + \frac{1}{2} \|\boldsymbol{\Sigma}_1 - \boldsymbol{\Sigma}_d\|_{\text{Frobenius}}^2$$

Prior work: Squared Wasserstein a.k.a. Optimal Transport Cost

$$\frac{1}{2} \|\boldsymbol{\mu}_1 - \boldsymbol{\mu}_d\|_2^2 + \frac{1}{2} \text{trace} \left(\boldsymbol{\Sigma}_1 + \boldsymbol{\Sigma}_d - 2 \left(\boldsymbol{\Sigma}_1^{\frac{1}{2}} \boldsymbol{\Sigma}_d \boldsymbol{\Sigma}_1^{\frac{1}{2}} \right)^{\frac{1}{2}} \right)$$

Halder and Wendel, ACC 2016

The Optimal Controller with Our Terminal Cost

From standard LQ theory: $\mathbf{u}_t^{\text{opt}} = \underbrace{\mathbf{K}_t^{\text{opt}} \mathbf{x}_t}_{\text{Feedback}} + \underbrace{\mathbf{v}_t^{\text{opt}}}_{\text{Feedforward}}$

The Optimal Controller with Our Terminal Cost

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Decoupling of mean and covariance control:

Feedforward gain

$$\mathbf{v}_t^{\text{opt}} = \mathbf{B}_t^\top \left(\mathbf{P}_t^{\text{opt}} \boldsymbol{\mu}_t^{\text{opt}} - \mathbf{z}_t^{\text{opt}} \right)$$

$$\begin{pmatrix} \dot{\boldsymbol{\mu}}_t^{\text{opt}} \\ \dot{\mathbf{z}}_t^{\text{opt}} \end{pmatrix} = \begin{bmatrix} \mathbf{A}_t & -\mathbf{B}_t \mathbf{B}_t^\top \\ -\mathbf{Q}_t & -\mathbf{A}_t^\top \end{bmatrix} \begin{pmatrix} \boldsymbol{\mu}_t^{\text{opt}} \\ \mathbf{z}_t^{\text{opt}} \end{pmatrix}$$

$\boldsymbol{\mu}_0 \in \mathbb{R}^n$ given, $\mathbf{z}_1^{\text{opt}} = \frac{\partial \phi}{\partial \boldsymbol{\mu}_1^{\text{opt}}} = \boldsymbol{\mu}_1^{\text{opt}} - \boldsymbol{\mu}_d$

$\mathbf{K}_t^{\text{opt}} = -\mathbf{B}_t^\top \mathbf{P}_t^{\text{opt}}$ **Feedback gain**

$$\dot{\boldsymbol{\Sigma}}_t^{\text{opt}} = \left(\mathbf{A}_t - \mathbf{B}_t \mathbf{B}_t^\top \mathbf{P}_t^{\text{opt}} \right) \boldsymbol{\Sigma}_t^{\text{opt}} + \boldsymbol{\Sigma}_t^{\text{opt}} \left(\mathbf{A}_t - \mathbf{B}_t \mathbf{B}_t^\top \mathbf{P}_t^{\text{opt}} \right)^\top + \mathbf{B}_t \mathbf{B}_t^\top$$

$$-\dot{\mathbf{P}}_t^{\text{opt}} = \mathbf{A}_t^\top \mathbf{P}_t^{\text{opt}} + \mathbf{P}_t^{\text{opt}} \mathbf{A}_t - \mathbf{P}_t^{\text{opt}} \mathbf{B}_t \mathbf{B}_t^\top \mathbf{P}_t^{\text{opt}} + \mathbf{Q}_t$$

$\boldsymbol{\Sigma}_0 \succ \mathbf{0}$ given,

$$\mathbf{P}_1^{\text{opt}} = \boldsymbol{\Sigma}_1^{\text{opt}} - \boldsymbol{\Sigma}_d, \boldsymbol{\Sigma}_d \succ \mathbf{0} \text{ given.}$$

Rest of this Talk

Focus on covariance control design:

$$\mathbf{K}_t^{\text{opt}} = -\mathbf{B}_t^\top \mathbf{P}_t^{\text{opt}} \quad \text{Feedback gain}$$

$$\dot{\Sigma}_t^{\text{opt}} = \left(\mathbf{A}_t - \mathbf{B}_t \mathbf{B}_t^\top \mathbf{P}_t^{\text{opt}} \right) \Sigma_t^{\text{opt}} + \Sigma_t^{\text{opt}} \left(\mathbf{A}_t - \mathbf{B}_t \mathbf{B}_t^\top \mathbf{P}_t^{\text{opt}} \right)^\top + \mathbf{B}_t \mathbf{B}_t^\top$$

$$-\dot{\mathbf{P}}_t^{\text{opt}} = \mathbf{A}_t^\top \mathbf{P}_t^{\text{opt}} + \mathbf{P}_t^{\text{opt}} \mathbf{A}_t - \mathbf{P}_t^{\text{opt}} \mathbf{B}_t \mathbf{B}_t^\top \mathbf{P}_t^{\text{opt}} + \mathbf{Q}_t$$

$$\Sigma_0 \succ \mathbf{0} \text{ given,}$$

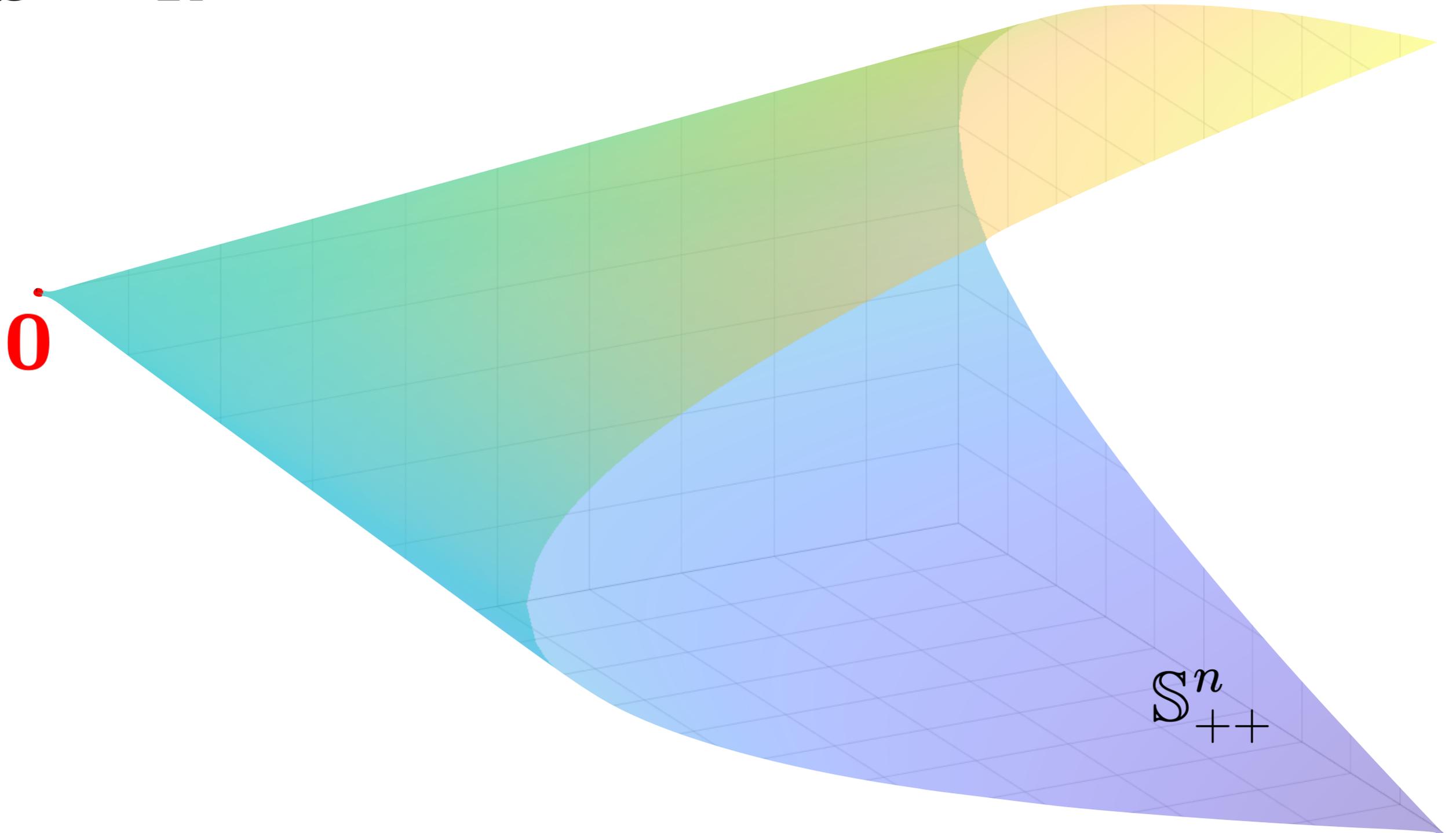
$$\mathbf{P}_1^{\text{opt}} = \Sigma_1^{\text{opt}} - \Sigma_d, \quad \Sigma_d \succ \mathbf{0} \text{ given.}$$

Coupled Lyapunov-Riccati nonlinear matrix ODE BVP

Our contribution: design numerical algorithm with guarantees to solve this system

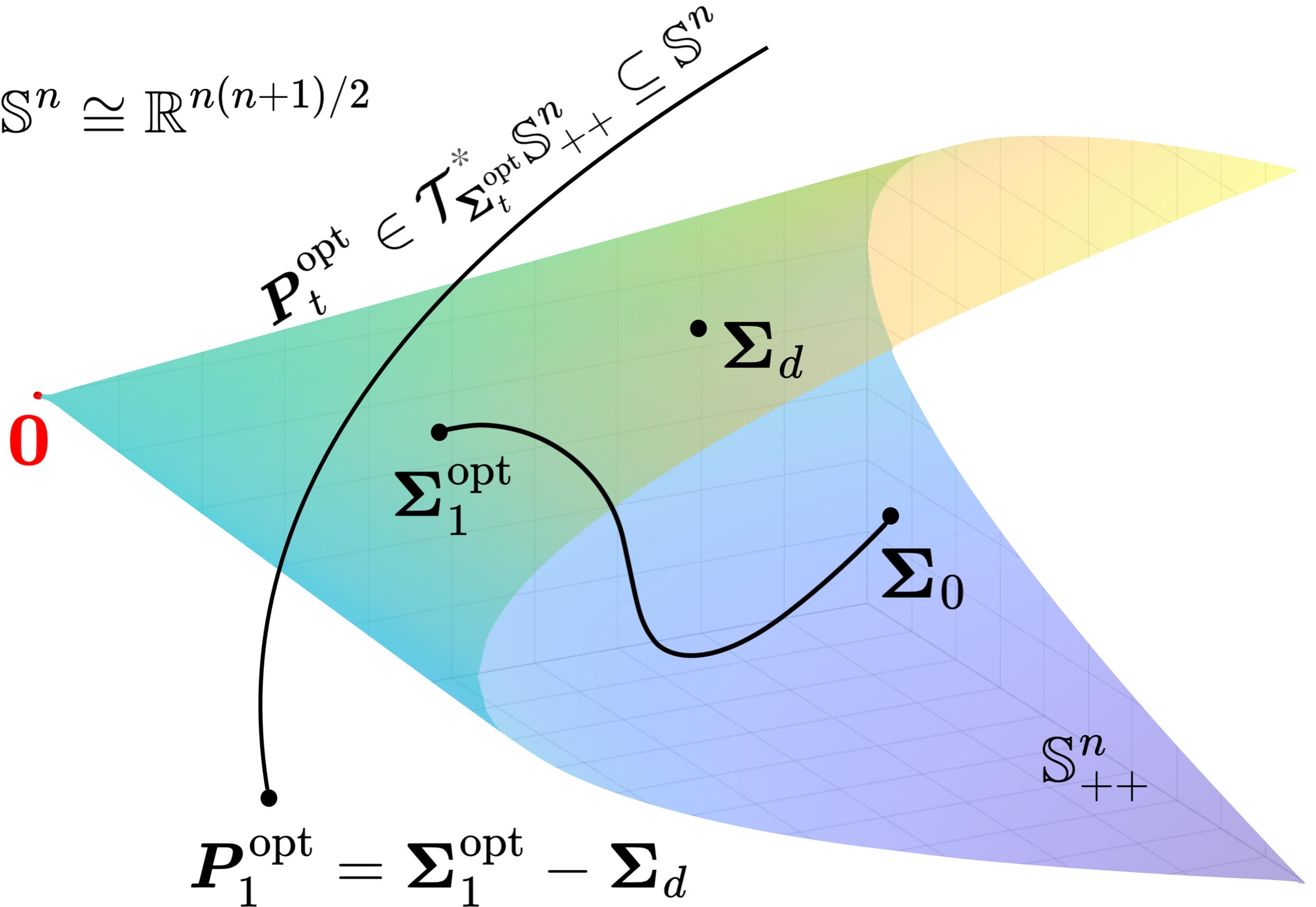
Geometry of Our Matrix ODE BVP

$$S^n \cong \mathbb{R}^{n(n+1)/2}$$



Geometry of Our Matrix ODE BVP

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$$P_t^{\text{opt}} \in \mathcal{T}_{\Sigma_t^{\text{opt}}}^* S^n_{++} \subseteq S^n$$

$$P_1^{\text{opt}} = \Sigma_1^{\text{opt}} - \Sigma_d$$

Idea

Change of variable $\Sigma_t \mapsto \mathbf{H}_t$

$$\mathbf{H}_t := \Sigma_t^{-1} - \mathbf{P}_t$$

From Lyapunov-Riccati to Riccati-Riccati system:

$$-\dot{\mathbf{P}}_t^{\text{opt}} = \mathbf{A}_t^\top \mathbf{P}_t^{\text{opt}} + \mathbf{P}_t^{\text{opt}} \mathbf{A}_t - \mathbf{P}_t^{\text{opt}} \mathbf{B}_t \mathbf{B}_t^\top \mathbf{P}_t^{\text{opt}} + \mathbf{Q}_t$$

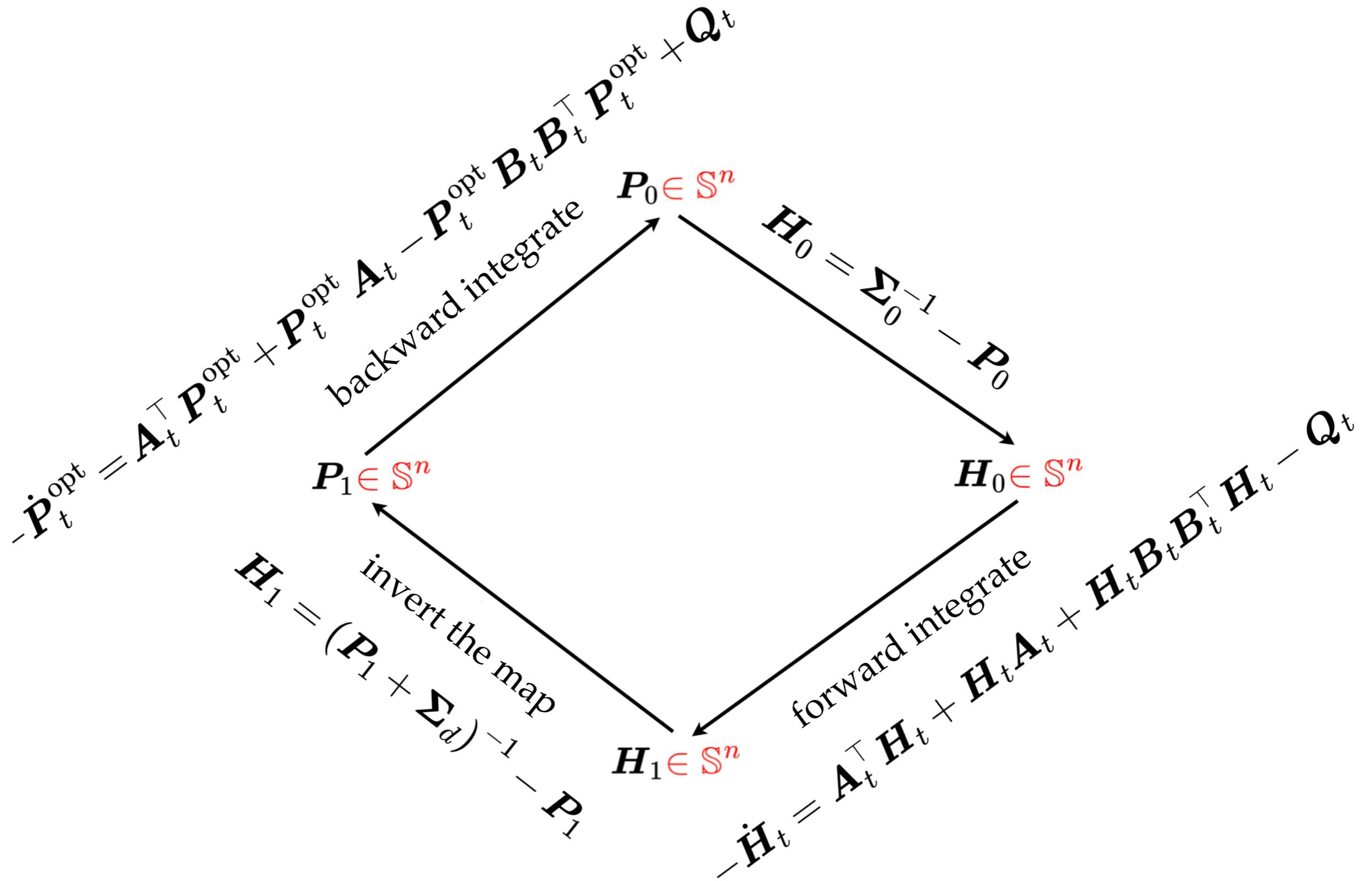
$$-\dot{\mathbf{H}}_t = \mathbf{A}_t^\top \mathbf{H}_t + \mathbf{H}_t \mathbf{A}_t + \mathbf{H}_t \mathbf{B}_t \mathbf{B}_t^\top \mathbf{H}_t - \mathbf{Q}_t$$

$$\mathbf{H}_0 = \Sigma_0^{-1} - \mathbf{P}_0$$

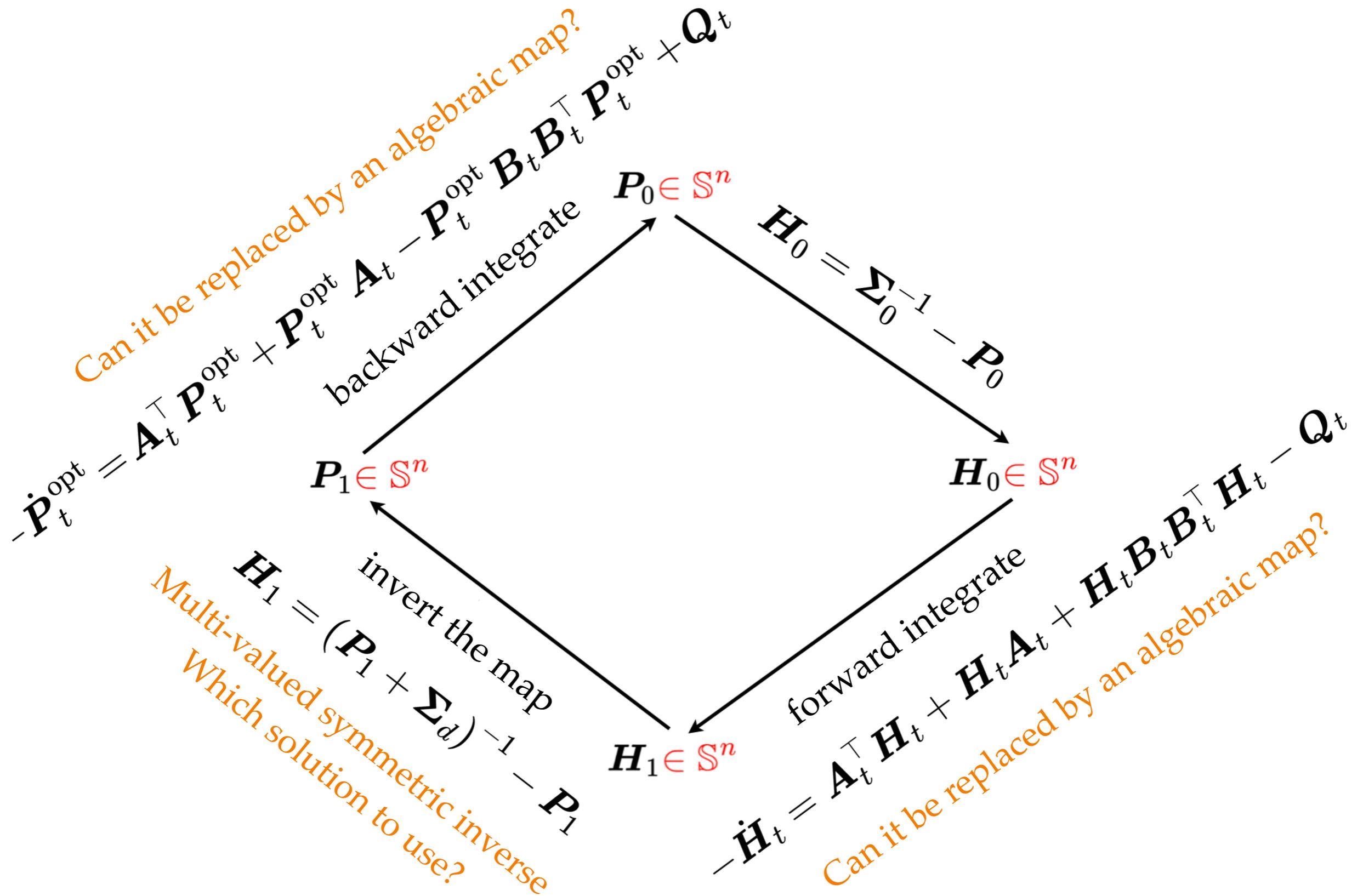
$$\mathbf{H}_1 = (\mathbf{P}_1 + \Sigma_d)^{-1} - \mathbf{P}_1$$

Achieves ODE-level decoupling but boundary conditions remain coupled

Proposed Fixed Point Recursion: First Cut



Proposed Fixed Point Recursion: First Cut



Mappings $P_1 \mapsto P_0, H_0 \mapsto H_1$

$$\text{Let } \mathbf{M}_t := \begin{bmatrix} \mathbf{A}_t & -\mathbf{B}_t \mathbf{B}_t^\top \\ -\mathbf{Q}_t & -\mathbf{A}_t^\top \end{bmatrix}$$

State transition matrix for this $2n \times 2n$ Hamiltonian matrix:

$$\partial_t \Phi(s, t) = \mathbf{M}(t) \Phi(s, t), \quad \Phi(s, s) = \mathbf{I}$$

Consider mild abuse of notation: $\begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} := \begin{bmatrix} \Phi_{11}(t_0, t_1) & \Phi_{12}(t_0, t_1) \\ \Phi_{21}(t_0, t_1) & \Phi_{22}(t_0, t_1) \end{bmatrix}$

Thm. Desired maps have bijective linear fractional transform representations:

$$P_0 = (P_1 \Phi_{12} - \Phi_{22})^{-1} (\Phi_{21} - P_1 \Phi_{11})$$

$$H_1 = -(\Phi_{11}^\top - H_0 \Phi_{12}^\top)^{-1} (\Phi_{21}^\top - H_0 \Phi_{22}^\top)$$

Mapping $H_1 \mapsto P_1$

We have

$$\mathbf{H}_1 = (\mathbf{P}_1 + \mathbf{\Sigma}_d)^{-1} - \mathbf{P}_1$$

This is equivalent to either of

$$\mathbf{I} = \mathbf{\Sigma}_1 (\mathbf{P}_1 + \mathbf{H}_1) = (\mathbf{P}_1 + \mathbf{\Sigma}_d) (\mathbf{P}_1 + \mathbf{H}_1)$$

$$\mathbf{I} = (\mathbf{P}_1 + \mathbf{H}_1) \mathbf{\Sigma}_1 = (\mathbf{P}_1 + \mathbf{H}_1) (\mathbf{P}_1 + \mathbf{\Sigma}_d)$$

Which is equivalent to either of

$$\mathbf{P}_1^2 + \mathbf{P}_1 \mathbf{H}_1 + \mathbf{\Sigma}_d \mathbf{P}_1 + (\mathbf{\Sigma}_d \mathbf{H}_1 - \mathbf{I}) = \mathbf{0}$$

$$\mathbf{P}_1^2 + \mathbf{P}_1 \mathbf{\Sigma}_d + \mathbf{H}_1 \mathbf{P}_1 + (\mathbf{H}_1 \mathbf{\Sigma}_d - \mathbf{I}) = \mathbf{0}$$

Effectively one equation in one symmetric unknown \mathbf{P}_1

But which of them or **equivalent of them** is suitable for our recursion?

Mapping $H_1 \mapsto P_1$

One natural equivalent is the quadratic equation:

$$P_1^2 + P_1 \left(\frac{H_1 + \Sigma_d}{2} \right) + \left(\frac{H_1 + \Sigma_d}{2} \right) P_1 + \left(\frac{\Sigma_d H_1 + H_1 \Sigma_d}{2} - I \right) = \mathbf{0}$$

This is a special instance of Continuous-time Algebraic Riccati Equation (CARE)

Thm. Given $H_1 \in \mathbb{S}^n$, $\Sigma_d \in \mathbb{S}_{++}^n$, there exists non-unique $P_1 \in \mathbb{S}^n$ solving above. Among them, the only stabilizing solution is

$$P_1 \equiv P_1^{\text{stab}} = -\frac{H_1 + \Sigma_d}{2} + \left(\left(\frac{H_1 - \Sigma_d}{2} \right)^2 + I \right)^{\frac{1}{2}}$$

We prove that for this solution $(P_1 + \Sigma_d)^{-1}$ exists

We also prove that this solution makes the proposed recursion converge!!

Mapping $H_1 \mapsto P_1$

Stabilizing (resp. anti-stabilizing) solution for CARE

$$P_1 \bar{B} P_1 + P_1 \bar{A} + \bar{A}^\top P_1 - \bar{C} = 0$$

is one that makes the closed-loop spectrum

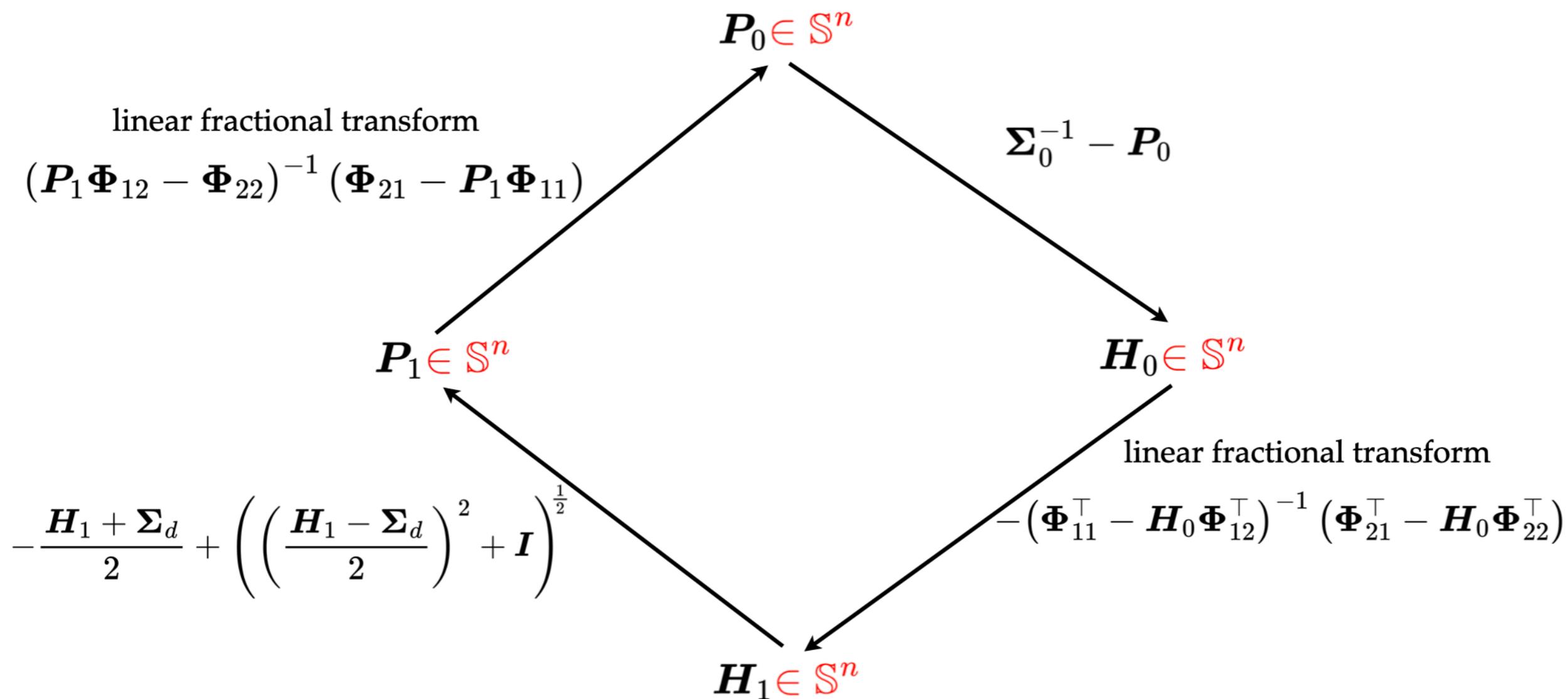
$$\text{spectrum} \left(-\bar{A} - \bar{B} \bar{B}^\top P_1 \right) \subset \mathbb{C}^- \quad (\text{resp. } \mathbb{C}^+)$$

All solutions $P_1 \in \mathbb{S}^n$ of our CARE satisfy the conic ordering:

$$P_1^{\text{antistab}} \preceq P_1 \preceq P_1^{\text{stab}}$$

In general, P_1^{antistab} , P_1^{stab} are sign-indefinite

Proposed Fixed Point Recursion: Final Cut



Proposed Fixed Point Recursion: Convergence

Thm. Let $F_1(\mathbf{X}) := \Sigma_0^{-1} - \mathbf{X}$,

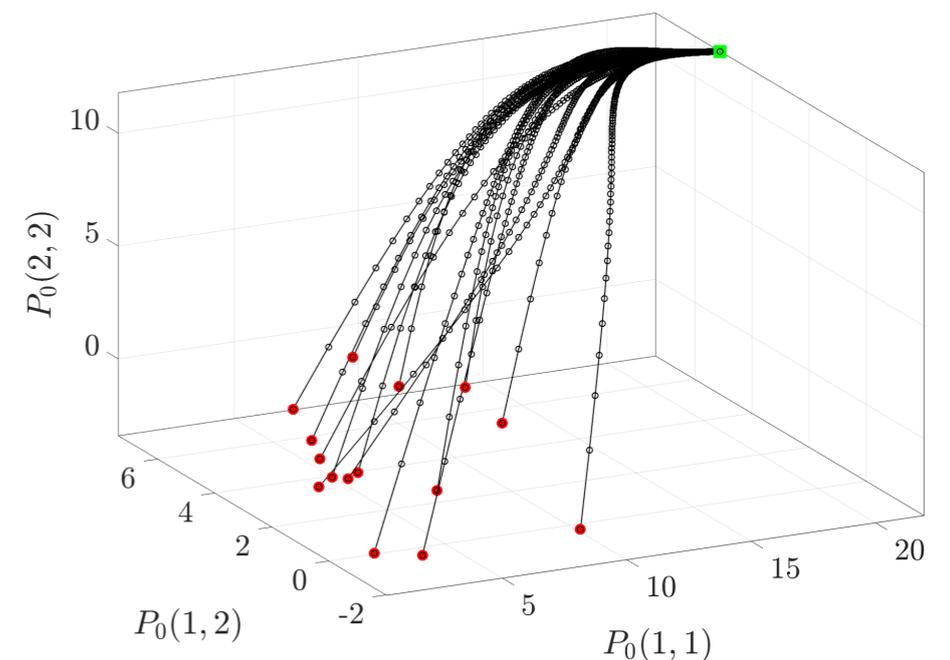
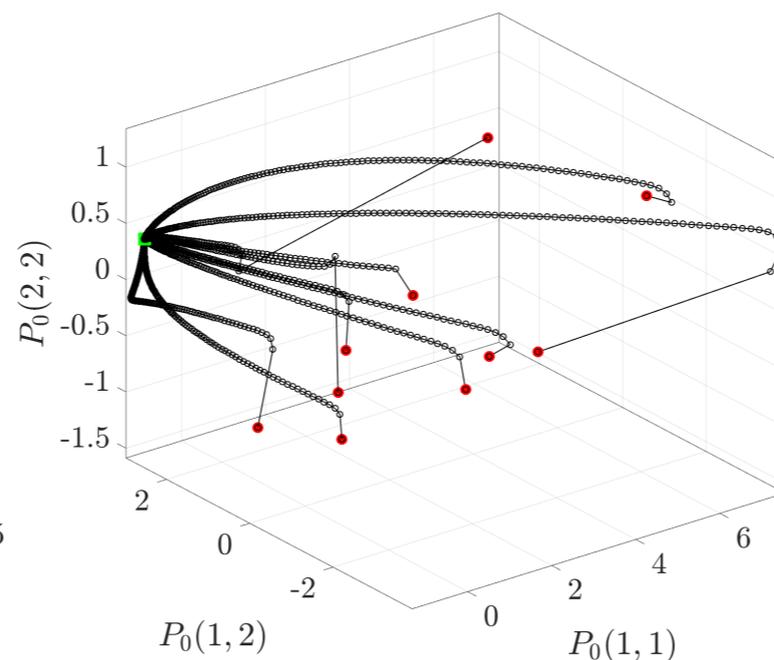
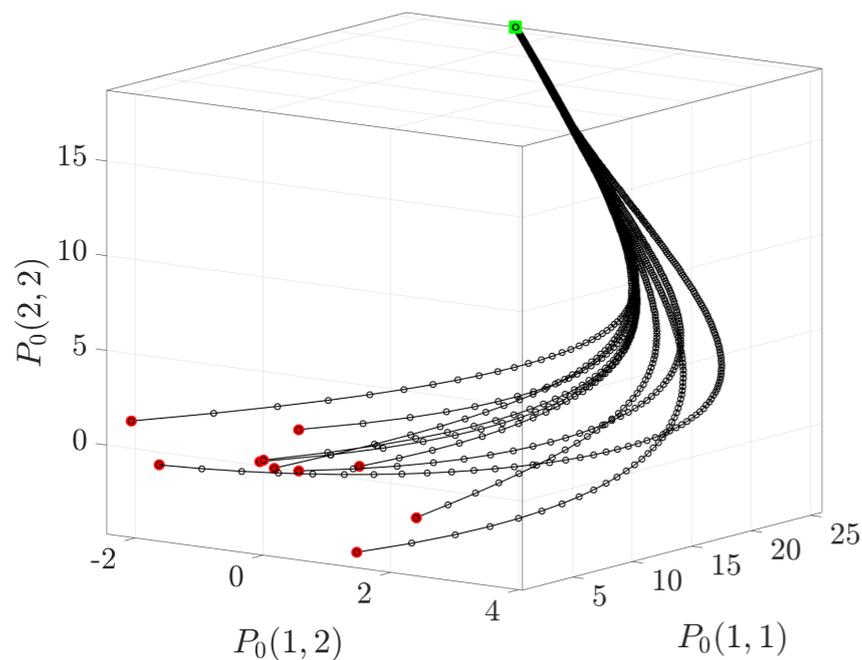
$$F_2(\mathbf{X}) := -(\Phi_{11}^\top - \mathbf{X}\Phi_{12}^\top)^{-1}(\Phi_{21}^\top - \mathbf{X}\Phi_{22}^\top)$$

$$F_3(\mathbf{X}) := -\frac{\mathbf{X} + \Sigma_d}{2} + \left(\left(\frac{\mathbf{X} - \Sigma_d}{2} \right)^2 + \mathbf{I} \right)^{\frac{1}{2}}$$

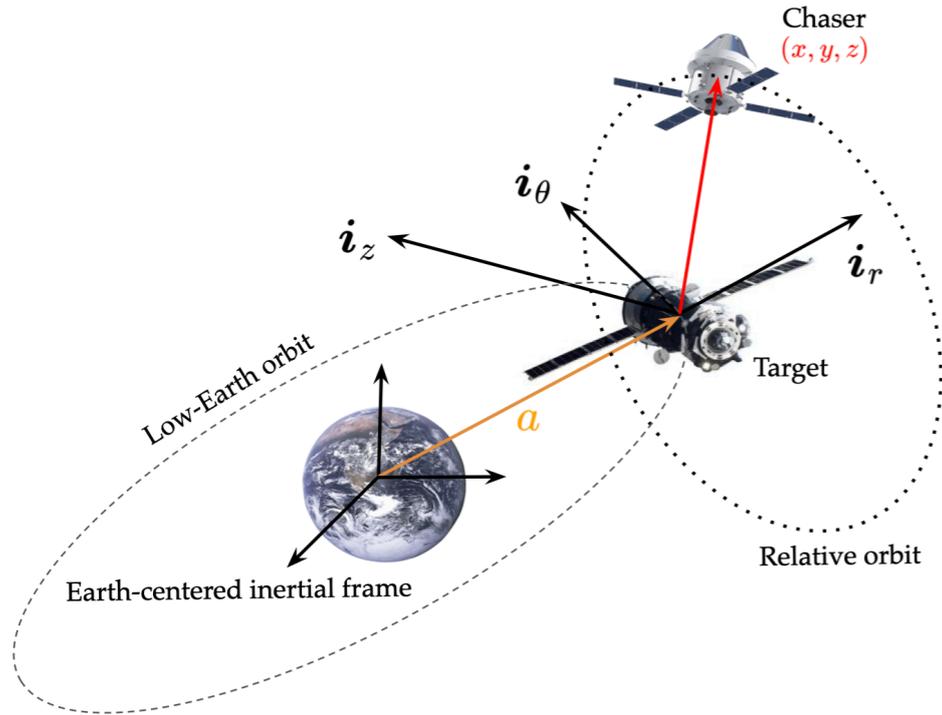
$$F_4(\mathbf{X}) := (\mathbf{X}\Phi_{12} - \Phi_{22})^{-1}(\Phi_{21} - \mathbf{X}\Phi_{11}),$$

in variable $\mathbf{X} \in \mathcal{S}^n$. Then $F := F_4 \circ F_3 \circ F_2 \circ F_1$ has unique fixed point in \mathcal{S}^n .

Also, the recursion $P_0 \mapsto (P_0)_{next} = F(P_0)$ converges to it for a.e. $P_0 \in \mathcal{S}^n$.



Numerical Case Study: Noisy Clohessy-Wiltshire



Controlled SDE:

$$dx_{1t} = x_{2t} dt,$$

$$dx_{2t} = (3\nu^2 x_{1t} + 2\nu x_{4t} + u_{1t}) dt + dw_{1t},$$

$$dx_{3t} = x_{4t} dt,$$

$$dx_{4t} = (-2\nu x_{2t} + u_{2t}) dt + dw_{2t},$$

$$dx_{5t} = x_{6t} dt,$$

$$dx_{6t} = (-\nu^2 x_{5t} + u_{3t}) dt + dw_{3t},$$

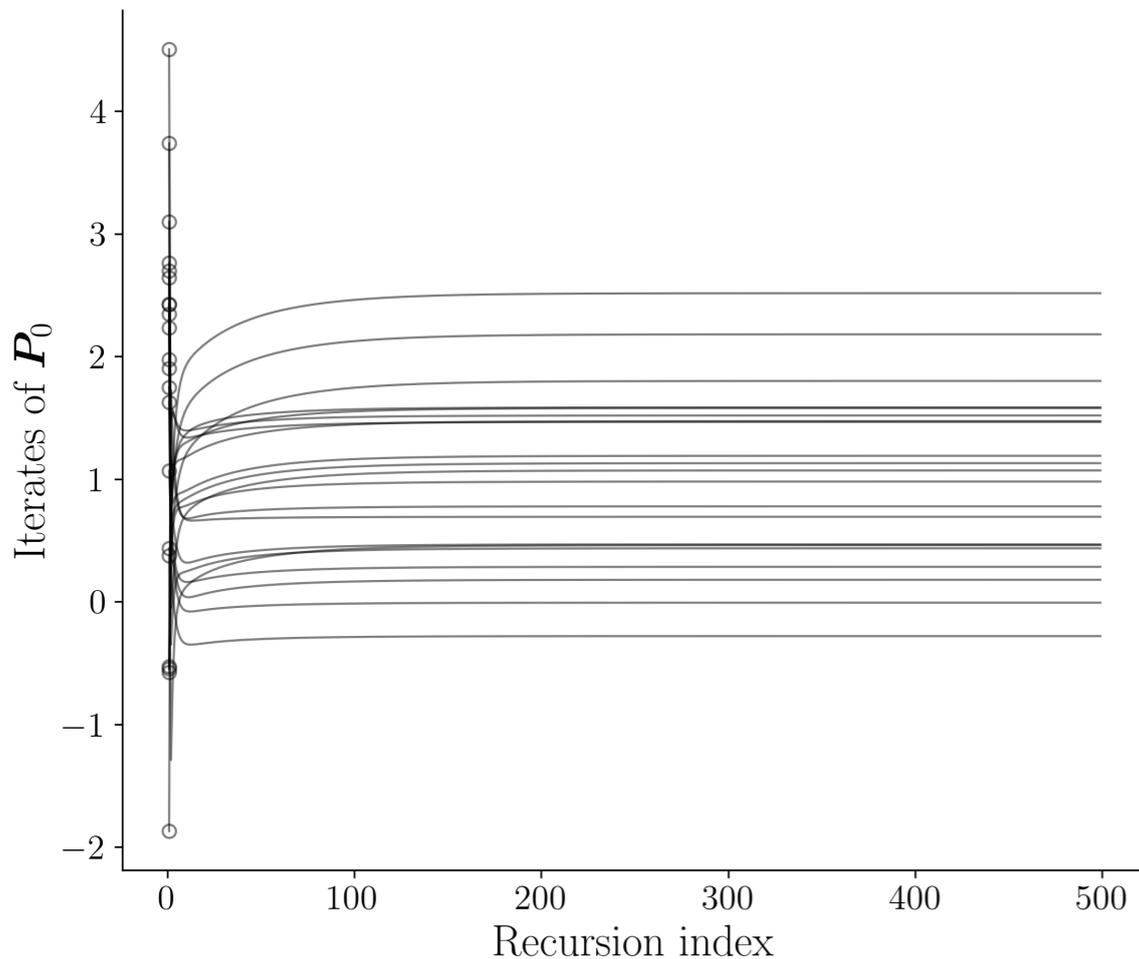
Parameters:

$$\nu := \sqrt{\mu/a^3} = 1.1276 \times 10^{-3} \text{ rad/s}$$

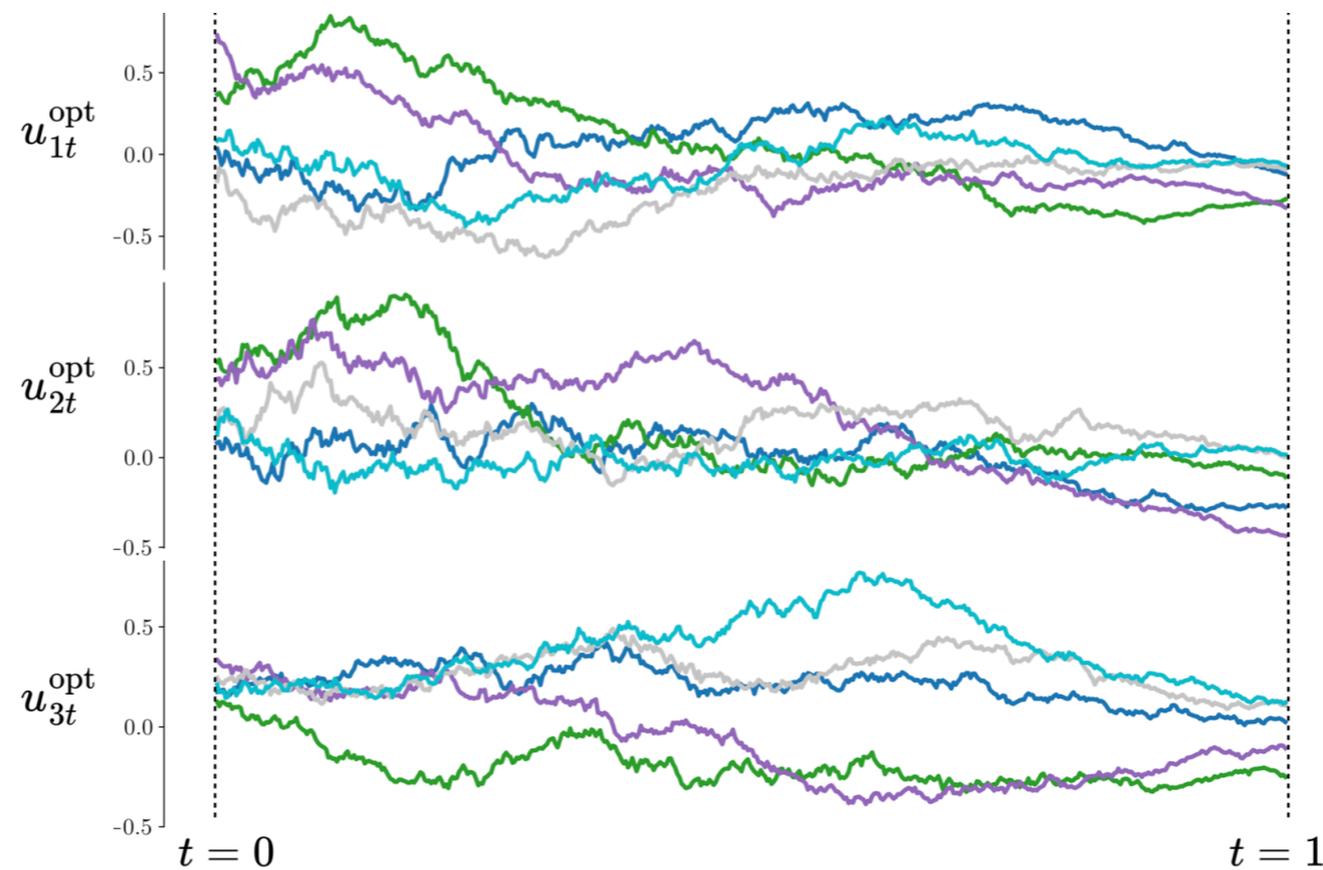
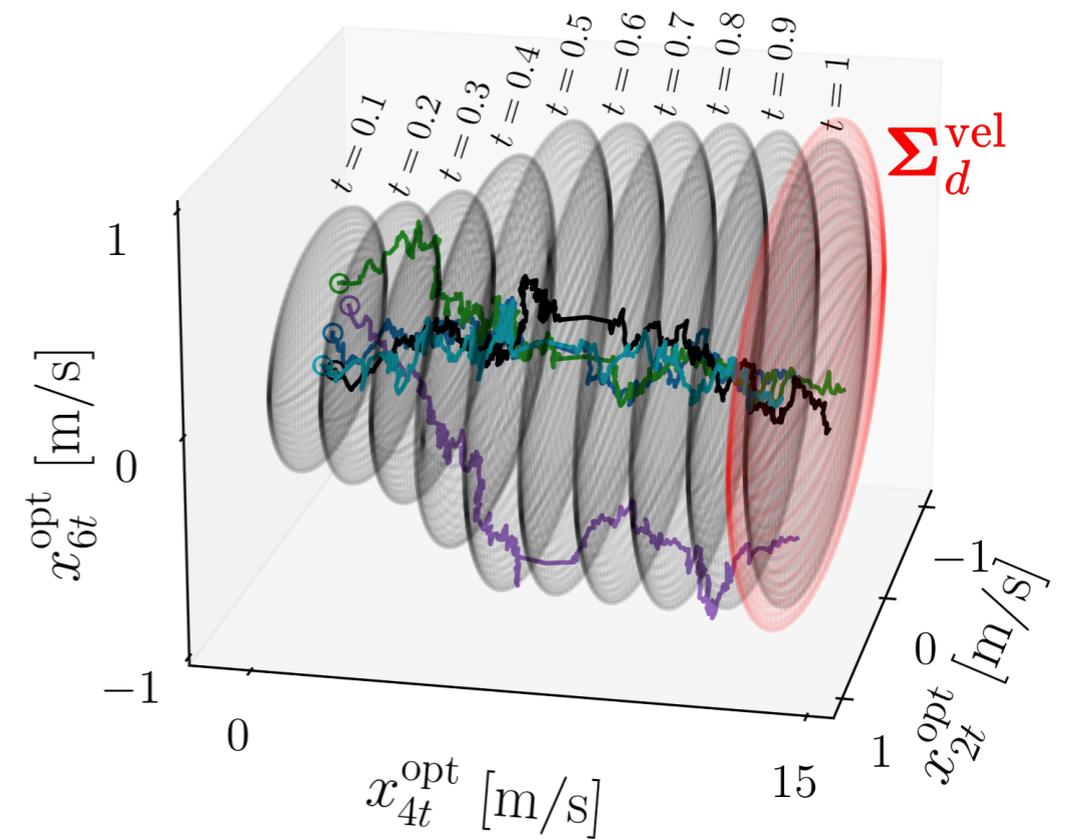
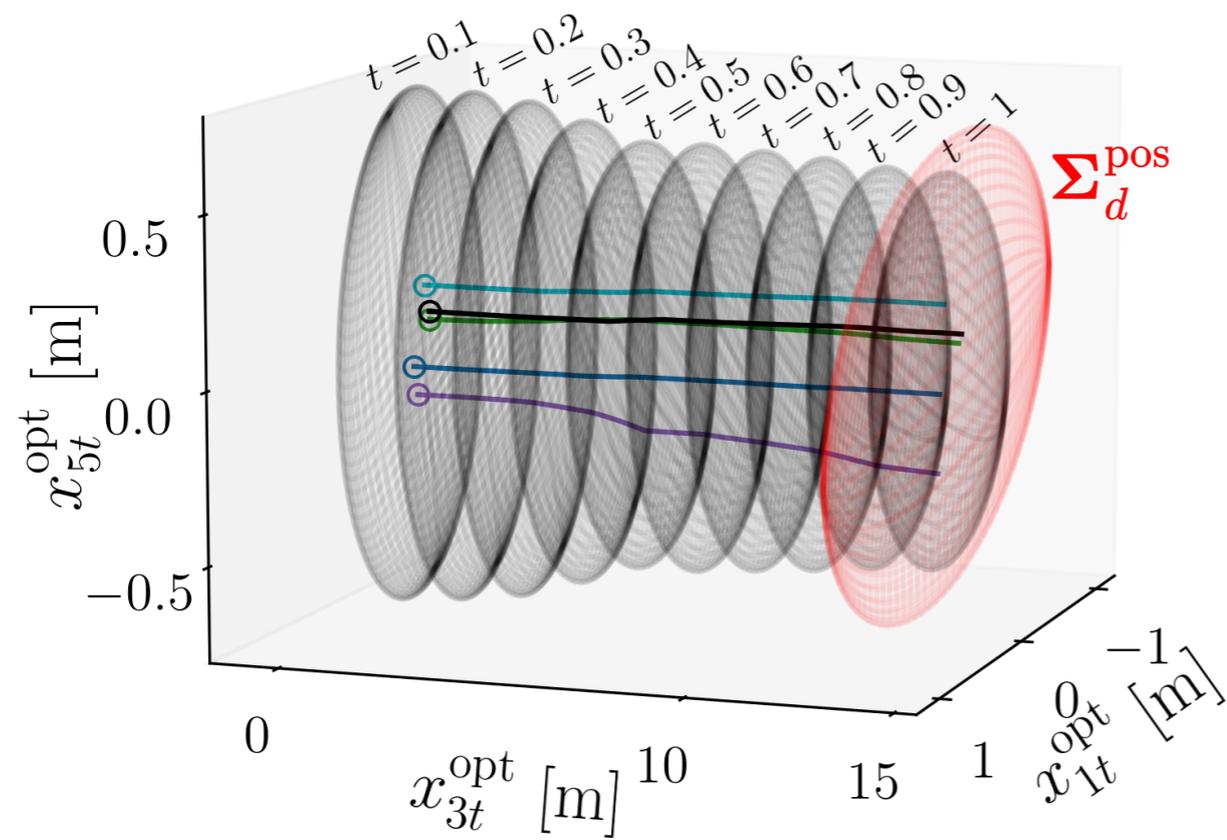
$$Q_t \equiv I,$$

$$\Sigma_0 = \begin{bmatrix} 5.9148 & 3.8100 & 2.5815 & 2.1795 & 4.1628 & 1.9270 \\ 3.8100 & 5.5664 & 2.8501 & 2.1819 & 3.8496 & 3.3638 \\ 2.5815 & 2.8501 & 3.3834 & 1.5591 & 2.5389 & 2.3088 \\ 2.1795 & 2.1819 & 1.5591 & 3.5850 & 2.6187 & 2.0098 \\ 4.1628 & 3.8496 & 2.5389 & 2.6187 & 5.1285 & 2.5639 \\ 1.9270 & 3.3638 & 2.3088 & 2.0098 & 2.5639 & 5.4354 \end{bmatrix}$$

$$\Sigma_d = \begin{bmatrix} 1.6431 & 1.1138 & 1.5453 & 1.1729 & 1.2916 & 0.4077 \\ 1.1138 & 1.9581 & 1.4418 & 1.0926 & 1.2408 & 0.4495 \\ 1.5453 & 1.4418 & 3.9142 & 1.9928 & 2.0221 & 1.5553 \\ 1.1729 & 1.0926 & 1.9928 & 2.1027 & 1.3448 & 0.9645 \\ 1.2916 & 1.2408 & 2.0221 & 1.3448 & 1.7077 & 0.7830 \\ 0.4077 & 0.4495 & 1.5553 & 0.9645 & 0.7830 & 1.5008 \end{bmatrix}$$



Numerical Case Study: Noisy Clohessy-Wiltshire



Summary

Advances the state-of-the-art for solving LQ covariance control with terminal cost

Proposes fixed point recursion with convergence guarantee

Illustrates practical computation using a numerical case study

Ongoing and Future Works

Optimal transport terminal cost:

$$\text{OT}(\boldsymbol{\Sigma}_1, \boldsymbol{\Sigma}_d) = \text{trace} \left(\boldsymbol{\Sigma}_1 + \boldsymbol{\Sigma}_d - 2 \left(\boldsymbol{\Sigma}_1^{\frac{1}{2}} \boldsymbol{\Sigma}_d \boldsymbol{\Sigma}_1^{\frac{1}{2}} \right)^{\frac{1}{2}} \right)$$

Bregman Divergence $D_\psi : \mathbb{S}_{++}^n \times \mathbb{S}_{++}^n \mapsto \mathbb{R}_{\geq 0}$ terminal costs:

Bregman divergence induced by mirror map $\psi : \mathbb{S}_{++}^n \mapsto \mathbb{R}$

$$D_\psi(\boldsymbol{\Sigma}_1, \boldsymbol{\Sigma}_d) := \psi(\boldsymbol{\Sigma}_1) - \left\{ \psi(\boldsymbol{\Sigma}_d) + \left\langle \frac{\partial \psi}{\partial \boldsymbol{\Sigma}_d}, \boldsymbol{\Sigma}_1 - \boldsymbol{\Sigma}_d \right\rangle \right\}$$

ψ	D_ψ	
$\ \mathbf{X}\ _F^2$	$\ \mathbf{X} - \mathbf{Y}\ _F^2$	This thesis
$\text{trace}(\mathbf{X} \log \mathbf{X} - \mathbf{X})$	$\text{trace}(\mathbf{X} \log \mathbf{X} - \mathbf{X} \log \mathbf{Y} - \mathbf{X} + \mathbf{Y})$	Quantum relative entropy
$-\log \det \mathbf{X}$	$\text{trace}(\mathbf{X}\mathbf{Y}^{-1}) - \log \det(\mathbf{X}\mathbf{Y}^{-1}) - n$	Kullback-Leibler divergence

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Thank You

Backup Slides

Recap

Frobenius a.k.a. Hilbert-Schmidt **inner product**

$$\langle \blacksquare, \blacksquare \rangle := \text{trace} (\blacksquare^\top \blacksquare) = \sum_{i,j} \blacksquare_{ij} \blacksquare_{ij}$$

Norm induced by this inner product

$$\langle \blacksquare, \blacksquare \rangle = \|\blacksquare\|^2$$

Generalizes squared Euclidean distance between vectors

Necessary Conditions of Optimality

We construct the Hamiltonian as follows:

$$\begin{aligned}
 H(\boldsymbol{\mu}_t, \mathbf{z}_t, \boldsymbol{\Sigma}_t, \mathbf{P}_t, \mathbf{K}_t, \mathbf{v}_t) &= \langle \mathbf{z}_t, (\mathbf{A}_t + \mathbf{B}_t \mathbf{K}_t) \boldsymbol{\mu}_t + \mathbf{B}_t \mathbf{v}_t \rangle \\
 &+ \langle \boldsymbol{\Sigma}_t + \boldsymbol{\mu}_t \boldsymbol{\mu}_t^\top, \mathbf{K}_t \mathbf{K}_t^\top + \mathbf{Q}_t \rangle + 2\mathbf{v}_t^\top \mathbf{K}_t \boldsymbol{\mu}_t + \mathbf{v}_t^\top \mathbf{v}_t \\
 &+ \langle \mathbf{P}_t, (\mathbf{A}_t + \mathbf{B}_t \mathbf{K}_t) \boldsymbol{\Sigma}_t + \boldsymbol{\Sigma}_t (\mathbf{A}_t + \mathbf{B}_t \mathbf{K}_t)^\top + \mathbf{B}_t \mathbf{B}_t^\top \rangle.
 \end{aligned}$$

Applying Pontryagin's minimum principle to the above Hamiltonian:

$$\dot{\boldsymbol{\mu}}_t^{\text{opt}} = \frac{\partial H}{\partial \mathbf{z}_t^{\text{opt}}}$$

$$\dot{\boldsymbol{\Sigma}}_t^{\text{opt}} = \frac{\partial H}{\partial \mathbf{P}_t^{\text{opt}}}$$

$$\dot{\mathbf{P}}_t^{\text{opt}} = -\frac{\partial H}{\partial \boldsymbol{\Sigma}_t^{\text{opt}}}$$

$$\dot{\mathbf{z}}_t^{\text{opt}} = -\frac{\partial H}{\partial \boldsymbol{\mu}_t^{\text{opt}}}$$

$$0 = \frac{\partial H}{\partial \mathbf{K}_t^{\text{opt}}} = (\mathbf{K}_t^{\text{opt}} + \mathbf{B}_t^\top \mathbf{P}_t^{\text{opt}}) \boldsymbol{\Sigma}_t^{\text{opt}}$$

$$0 = \frac{\partial H}{\partial \mathbf{v}_t^{\text{opt}}} = (\mathbf{v}_t^{\text{opt}} - \mathbf{B}_t^\top \mathbf{P}_t^{\text{opt}} \boldsymbol{\mu}_t^{\text{opt}} + \mathbf{B}_t^\top \mathbf{z}_t^{\text{opt}})$$

$$\mathbf{z}_1^{\text{opt}} = \frac{\partial \phi}{\partial \boldsymbol{\mu}_1^{\text{opt}}}$$

$$\mathbf{P}_1^{\text{opt}} = \frac{\partial \phi}{\partial \boldsymbol{\Sigma}_1^{\text{opt}}}$$

Properties of State Transition Matrix

The blocks of the state transition matrix satisfy the following identities:

$$\Phi_{11}^\top \Phi_{22} - \Phi_{21}^\top \Phi_{12} = \mathbf{I}$$

$$\Phi_{12}^\top \Phi_{22} - \Phi_{22}^\top \Phi_{12} = \mathbf{0}$$

$$\Phi_{21}^\top \Phi_{11} - \Phi_{11}^\top \Phi_{21} = \mathbf{0}$$

$$\Phi_{11} \Phi_{22}^\top - \Phi_{12} \Phi_{21}^\top = \mathbf{I}$$

$$\Phi_{12} \Phi_{11}^\top - \Phi_{11} \Phi_{12}^\top = \mathbf{0}$$

$$\Phi_{21} \Phi_{22}^\top - \Phi_{22} \Phi_{21}^\top = \mathbf{0}$$

Furthermore, Φ_{11} and Φ_{12} are invertible.

Mapping $H_1 \mapsto P_1$

One natural equivalent is linear (Sylvester's equation):

$$(\mathbf{H}_1 - \mathbf{\Sigma}_d)\mathbf{P}_1 + \mathbf{P}_1(\mathbf{\Sigma}_d - \mathbf{H}_1) = \mathbf{\Sigma}_d\mathbf{H}_1 - \mathbf{H}_1\mathbf{\Sigma}_d$$

Thm. Given $\mathbf{H}_1 \in \mathbb{S}^n$, $\mathbf{\Sigma}_d \in \mathbb{S}_{++}^n$, there exists non-unique $\mathbf{P}_1 \in \mathbb{S}^n$ solving above

Valid candidate solutions:

$-\mathbf{H}_1, -\mathbf{\Sigma}_d$, solution of the linear system:

$$((\mathbf{I} \otimes (\mathbf{H}_1 - \mathbf{\Sigma}_d)) - ((\mathbf{H}_1 - \mathbf{\Sigma}_d) \otimes \mathbf{I}))\text{vec}(\mathbf{P}_1) = \text{vec}(\mathbf{\Sigma}_d\mathbf{H}_1 - \mathbf{H}_1\mathbf{\Sigma}_d)$$

But for these solutions, $(\mathbf{P}_1 + \mathbf{\Sigma}_d)^{-1}$ may not exist

They also fail to make proposed recursion converge!!

Mapping $H_1 \mapsto P_1$ (Contd.)

Stabilizing (resp. anti-stabilizing) solution for CARE

$$P_1 \bar{B} P_1 + P_1 \bar{A} + \bar{A}^\top P_1 - \bar{C} = \mathbf{0}$$

is one that makes the closed-loop spectrum

$$\text{spectrum} \left(-\bar{A} - \bar{B} \bar{B}^\top P_1 \right) \subset \mathbb{C}^- \quad (\text{resp. } \mathbb{C}^+)$$

for our CARE, the LTI pair is

$$(\bar{A}, \bar{B}) := \left(\frac{H_1 + \Sigma_d}{2}, I \right)$$

Numerical Case Study: Noisy Double Integrator

Controlled SDE:

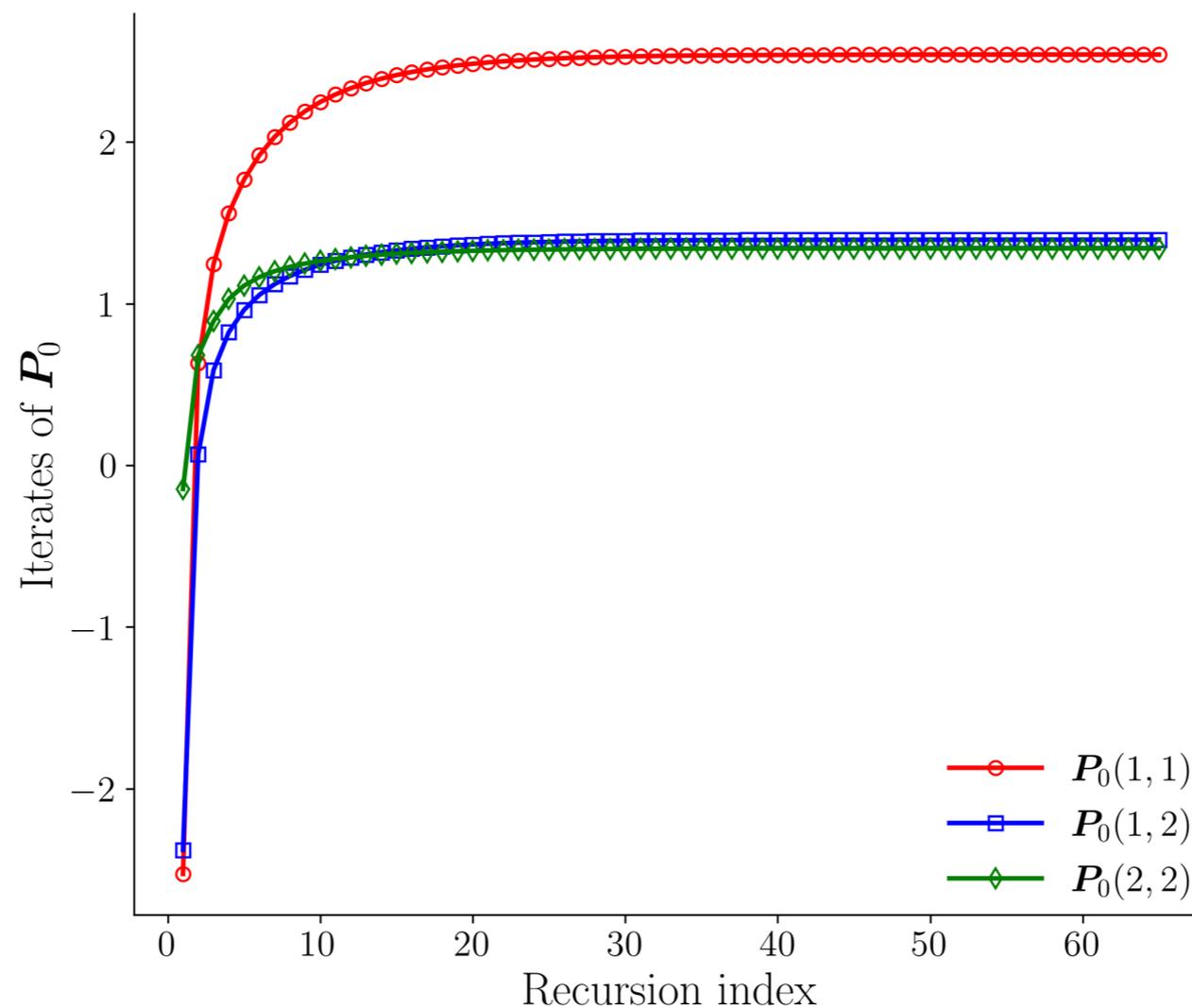
$$dx_{1t} = x_{2t} dt,$$

$$dx_{2t} = u_t dt + dw_t$$

Parameters: $Q_t \equiv I,$

$$\Sigma_0 = \begin{bmatrix} 4.7295 & 1.9951 \\ 1.9951 & 3.6157 \end{bmatrix},$$

$$\Sigma_d = \begin{bmatrix} 1.1189 & 0.7780 \\ 0.7780 & 1.7407 \end{bmatrix}$$



Numerical Case Study: Noisy Double Integrator

